The Equivalents of Axiom of Choice

1. **Axiom of Choice.** The Cartesian product of a nonempty family of nonempty sets is nonempty.

2. **Choice Function for Subsets.** Let X be a nonempty set. Then for each nonempty subset $S \subseteq X$ it is possible to choose some element $s \in S$. That is, there exists a function f which assigns to each nonempty set $S \subseteq X$ some representative element $f(S) \in S$.

3. **Set of Representatives.** Let $\{X_{\boldsymbol{l}} : \boldsymbol{l} \in \boldsymbol{L}\}$ be a nonempty set of nonempty sets which are pairwise disjoint. Then there exists a set C containing exactly one element from each $X_{\boldsymbol{l}}$.

4. **Nonempty Products.** If $\{X_{\boldsymbol{l}} : \boldsymbol{l} \in \boldsymbol{L}\}$ is a nonempty set of nonempty sets, then the Cartesian product $\prod_{\boldsymbol{l} \in \boldsymbol{L}} X_{\boldsymbol{l}}$ is nonempty. That is, there exists a function $f : \boldsymbol{L} \to \bigcup_{\boldsymbol{l} \in \boldsymbol{L}} X_{\boldsymbol{l}}$ satisfying $f(\boldsymbol{l}) \in X_{\boldsymbol{l}}$ for each $\boldsymbol{l}$.

5. **Well-Ordering Principle (Zermelo).** Every set can be well ordered.

6. **Finite Character Principle (Tukey, Teichmuller).** Let X be a set, and let F be a collection of subsets of X. Suppose that F has finite character (i.e., a set is a member of F if and only if each finite subset of that set is a member of F). Then any member of F is a subset of some $\subseteq$-maximal member of F.

7. **Maximal Chain Principle (Hausdorff).** Let $(X, \preceq)$ be a partially ordered set. Then any $\preceq$-chain in X is included in a $\subseteq$-maximal $\preceq$-chain.

8. **Zorn's Lemma (Hausdorff, Kuratowski, Zorn, others).** Let $(X, \preceq)$ be a partially ordered set. Assume every $\preceq$-chain in X has a $\preceq$-upper bound in X. Then X has a $\preceq$-maximal element.

9. **Weakened Zorn's Lemma.** Let $(X, \preceq)$ be a partially ordered set. Assume every subset of X that is directed by $\preceq$ has a $\preceq$-upper bound in X. Then X has a $\preceq$-maximal element.

10. **Well-Ordering of Cardinals.** Comparison of cardinalities is a well ordering. That is: If Σ is a set whose elements are sets, then there is some $S_0 \in \Sigma$ which satisfies $S_0 \in |T|$ for all $T \in \Sigma$.

11. **Trichotomy of Cardinals.** Comparison of cardinalities is a chain ordering. That is, for any two sets S and T, precisely one of these three conditions holds: $|S| < |T|$; $|S| = |T|$; $|S| > |T|$.

12. **Comparability of Hartogs Number.** Let $H(S)$ is the Hartogs number of a set S —i.e., the first ordinal that does not have cardinality less than or equal to $|S|$. Then $|H(S)|$ and $|S|$ are comparable —i.e., one is bigger than or equal to the other (and hence $|H(S)| > |S|$).

13. **Squaring of Cardinals.** If X is an infinite set then $|X \times X| = |X|$.

14. **Multiplication of Cardinals.** If X is an infinite set, Y is a nonempty set, and $|X| \geq |Y|$, then $|X \cup Y| = |X|$.

15. **Another cardinality result.** If X and Y are disjoint sets, $|X| > |\mathrm{N}|$, and Y is nonempty, then $|X \cup Y| = |X \times Y|$.

16. **Another cardinality result.** If X is an infinite set, then the cardinality of X is equal to the cardinality of $\bigcup_{n=1}^{\infty} X^n = \{\text{finite sequences in } X\}$.

17. **Vector Basis Theorem (Strong Form).** Let X be a linear space over some field. Suppose that I is a linearly independent subset of X, G is generating set (that is, span(G) = X), and $I \subseteq G$. Then $I \subseteq B \subseteq G$ for some vector basis B.

18. **Vector Basis Theorem (Intermediate Form).** Let X be a linear space over some field and let G be a subset of X which generates X (that is, span(G) = X). Then X has a vector basis B contained in G.

19. **Vector Basis Theorem (Weak Form).** Any linear space over any field has a vector basis over that field.

20. **Product of Closures.** For each l in some index set L, let S_l be a subset of some topological space X_l. Then $\mathrm{Cl}\left(\prod_{l \in L} S_l\right) = \left(\prod_{l \in L} \mathrm{Cl}(S_l)\right)$.

21. **Product of Closures (Weak Form).** $\mathrm{Cl}\left(\prod_{l \in L} S_l\right) \supseteq \left(\prod_{l \in L} \mathrm{Cl}(S_l)\right)$.

22. **Tychonoff Product Theorem.** Any product of compact topological spaces is compact.

Reference:

1. Schechter Eric, "*Handbook of Analysis and Its Foundations*", Academic Press, San Diego, c1997.